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A two-dimensional Boussinesq equation for water waves and some of its solutions

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A two-dimensional Boussinesq equation,

$$u_{tt} - u_{xx} + 3(u^2)_{xx} - u_{xxxx} - u_{yy} = 0,$$

is introduced to describe the propagation of gravity waves on the surface of water, in particular the head-on collision of oblique waves. This equation combines the two-way propagation of the classical Boussinesq equation with the (weak) dependence on a second spatial variable, as occurs in the two-dimensional Korteweg–de Vries (2D KdV) (or KPII) equation. Exact and general solitary-wave, two-soliton and resonant solutions are obtained from the Hirota bilinear form of the equation. The existence of a distributed-soliton solution is investigated, but it is shown that this is not a possibility. However the connection with the classical 2D KdV equation (which does possess such a solution) is explored via a suitable parametric representation of the dispersion relation.

A three-soliton solution is also constructed, but this exists only if an auxiliary constraint among the six parameters is satisfied; thus the two-dimensional Boussinesq equation is not one of the class of completely integrable equations, confirming the analysis of Hietarinta (1987). This constraint is automatically satisfied for the classical Boussinesq equation (which is completely integrable). Graphical reproductions of some of the solutions of the two-dimensional Boussinesq equations are also presented.

1. Introduction

Completely integrable equations, that is, evolution equations of soliton type, have been with us now for almost thirty years. They appear, and are important and relevant, in many different branches of applied mathematics and physics; not least, many examples and variants are generated by problems in the study of water waves, which is the vehicle we use here. Indeed, the archetypal equation – the Korteweg–de Vries (KdV) equation – first saw the light of day in work on long gravity waves moving over stationary water (Korteweg & de Vries 1895); this equation, written in the familiar normalized version, is

$$u_t - 6uu_x + u_{xxx} = 0, \quad t > 0, \quad -\infty < x < \infty. \tag{1}$$

Once the general method of solution of this equation was obtained (in the seminal work of Gardner *et al.* 1967), many other equations and mathematical approaches rapidly followed. In the particular field of water waves, two families of nonlinear evolution equations emerge: one is the KdV family of equations, and the other is based on the nonlinear Schrödinger (NLS) equation,

$$\mathrm{i}u_t + u_{xx} + u|u|^2 = 0.$$

Here we shall be concerned only with the KdV family and, specifically, with a new member of this family which would appear to be of some relevance in water-wave theory (as well as being an equation which can be written down following simple rules, and which may be worthy of study in its own right).

Under appropriate assumptions that describe the requirements for small amplitude and long waves (which we shall define more carefully later), one-dimensional surface gravity waves satisfy the KdV equation, (1), to leading order in a suitable region of physical space. This KdV equation is relevant to waves that propagate only in one direction. If the waves propagate *mainly* in one direction, but oblique interactions or slightly bent wave fronts are allowed, the governing equation (under suitable assumptions) turns out to be of the form

$$(u_t - 6uu_x + u_{xxx})_x + u_{yy} = 0, \tag{2}$$

the Kadomtsev–Petviashvili (KP), or 2D KdV, equation; Kadomtsev & Petviashvili (1970). The KP equations – there are two of them, depending on the sign of the term in u_{yy} (ours is KPII) – have excited much interest because they are the archetype of completely integrable equations in $2+1$ dimensions; see Ablowitz & Clarkson (1991) for an excellent introduction to much of this work. It is also possible to have concentric (or nearly concentric) waves of this type; thus, for example, we can obtain

$$2u_t + \frac{1}{t}u - 6uu_x + u_{xxx} = 0 \tag{3}$$

for concentric or cylindrical waves. (This, and other equations in the KdV family for water waves, are discussed in Johnson 1980.) All these examples relate to waves that are moving, predominantly, in one direction – to the left or right, or outwards or inwards. However, another equation, similar in character to the KdV equation, describes waves that are moving in one dimension but which may propagate in *opposite* directions. This equation, usually written in the form

$$u_{tt} - u_{xx} + 3(u^2)_{xx} - u_{xxxx} = 0, \tag{4}$$

admits solutions in which the waves may collide head-on; all the other equations mentioned earlier permit only overtaking collisions; equation (4) is called the Boussinesq equation (Hirota 1973). Thus we refer to equations (1)–(4), and some others not written down here, as belonging to the KdV family, and one aim of this work is to extend this family (albeit in a rather obvious way).

Equation (2) is the two-dimensional extension of the classical KdV equation (1), where the dependence on the second dimension (through y) is suitably weak; we shall describe this requirement below. The question we pose here is whether a similar argument can be used for the Boussinesq equation, and hence produce a two-dimensional version of that equation relevant to water waves. Such an equation, presumably, will then admit solutions that represent, for example, *head-on* collisions between *oblique* waves. We shall briefly describe the derivation of this new equation, which we shall call the two-dimensional Boussinesq equation, and which takes – perhaps not surprisingly – the form

$$u_{tt} - u_{xx} + 3(u^2)_{xx} - u_{xxxx} - u_{yy} = 0. \tag{5}$$

This equation, (5), would seem to be the archetype for waves that propagate in *opposite* directions in $2+1$ dimensions.

In the context of evolution equations, an important question is whether a given equation is one of the completely integrable class; equations (1)–(4) belong to this class. Now Hietarinta (1987) has generated, via the bilinear form, those equations which belong to the KdV family and which also are completely integrable. (In later papers he considered equations that belong to other families.) Our two-dimensional Boussinesq equation is not on Hietarinta's list, and so we must expect that it will not be completely integrable. Certainly, if our equation can be expressed in bilinear form, then Hietarinta (1987) has shown that it must possess at least a general two-soliton solution. The new equation, as we shall demonstrate, can be written in bilinear form. The crucial test, as Hietarinta explains, is whether a general three-soliton solution exists (for this is related to the *Painlevé property*, which is discussed, for example, in Ablowitz & Clarkson 1991). We shall derive the two-soliton solution of the two-dimensional Boussinesq equation, and also show that a general three-soliton solution does not exist, confirming the predictions of Hietarinta (1987). This might be regarded as the end of the story, and it would be if we are driven by the need to study only completely integrable equations (and then we would find that much of the work is routine). Our new equation is relevant to water waves and so, we submit, is worthy of study – but now we cannot call on the full panoply of soliton theory.

Thus, because of the rôle of this equation in water-wave theory, we shall examine some solutions and properties that are peculiar to it in this respect. We shall, for example, describe the special two-soliton solutions which are resonant-wave interactions (Miles 1977; Freeman 1980). Also we show how the two-dimensional Boussinesq (very simply) goes over to the classical Boussinesq equation (which is completely integrable) and, more significantly, we investigate the existence of distributed soliton solutions and relate this to the corresponding solutions of the 2D KdV (KPII) equation; see Freeman (1979). Extensive numerical solutions, relevant to water waves, are left for a later study.

2. Governing equations

We consider wave propagation on the surface of water (taken to be an inviscid fluid) of constant depth, which is stationary in its undisturbed state; the surface-tension effects are ignored. The governing equations are therefore Euler's equation – this is regarded as marginally preferable to Laplace's equation – together with the appropriate surface and bottom boundary conditions. The problem is non-dimensionalized using the undisturbed depth of water h, a typical wavelength of the wave λ, and a typical amplitude of the wave a; see figure 1. The non-dimensional equations are then written in terms of two parameters: $\epsilon = a/h$, the amplitude parameter, and $\delta = h/\lambda$, the shallowness parameter. It turns out to be slightly more convenient to redefine the problem by scaling out the parameter δ (which equivalently replaces δ^2 by ϵ), which we may always do. Thus we transform, for example,

$$x \rightarrow \frac{\delta}{\epsilon^{1/2}} x,$$

and correspondingly for t, y and w; the resulting (non-dimensional) equations are then

$$\left.\begin{aligned} u_t + \epsilon(uu_x + vu_y + wu_z) &= -p_x, \\ v_t + \epsilon(uv_x + vv_y + wv_z) &= -p_y, \\ \epsilon\{w_t + \epsilon(uw_x + vw_y + ww_z)\} &= -p_z, \end{aligned}\right\} \tag{6}$$

with

$$u_x + v_y + w_z = 0, \tag{7}$$

and the boundary conditions are

$$p = \eta, \quad w = \eta_t + \epsilon(u\eta_x + v\eta_y) \quad \text{on} \quad z = 1 + \epsilon\eta; \tag{8}$$

$$w = 0 \quad \text{on} \quad z = 0. \tag{9}$$

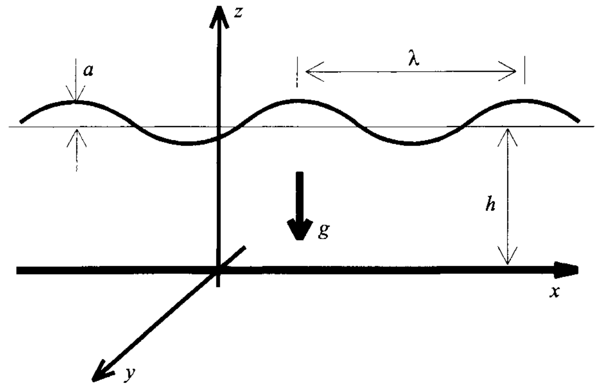

FIGURE 1. Defining sketch for the variables and scales used in the water–wave problem.

Here, the surface has been written as $z = 1+\epsilon\eta(x,y,t;\epsilon)$ and p is measured relative to the hydrostatic pressure in the undisturbed state.

If we choose to consider strictly one-dimensional waves (so $v \equiv 0$, $\partial/\partial y \equiv 0$), and we introduce the far-field variables

$$\xi = x-t, \quad \tau = \epsilon t,$$

an asymptotic solution in integer powers of ϵ, as $\epsilon \to 0$, yields the KdV equation

$$2\eta_{0\tau}+3\eta_0\eta_{0\xi}+\tfrac{1}{3}\eta_{0\xi\xi\xi} = 0. \tag{10}$$

We have written $\eta \sim \eta_0(\xi,\tau)$, as $\epsilon \to 0$, and the choice of the characteristic variable $x-t$ (rather than $x+t$) shows that we are following right-running waves.

Now, if we seek a solution which also depends (weakly) on the y-coordinate, by introducing $Y = \epsilon^{1/2}y = O(1)$ (and then we require $v = \epsilon^{1/2}V$), we find that $\eta \sim \eta_0(\xi,\tau,Y)$ satisfies the 2D KdV (KPII) equation

$$(2\eta_{0\tau}+3\eta_0\eta_{0\xi}+\tfrac{1}{3}\eta_{0\xi\xi\xi})_\xi+\eta_{0YY} = 0. \tag{11}$$

Finally, if we allow waves to propagate in either direction (so we must work with the original x and t), and include the weak dependence on y (as above), then we obtain

$$\eta_{tt}-\eta_{xx}-\epsilon\left\{\tfrac{1}{2}\eta^2+\left(\int_x^\infty \eta_t\,\mathrm{d}x\right)^2\right\}_{xx}-\frac{\epsilon}{3}\eta_{xxxx}+\epsilon V_{Yt} = O(\epsilon^2), \tag{12}$$

where $V_t = -\eta_Y+O(\epsilon)$. This equation is more usually written in terms of

$$H = \eta-\epsilon\eta^2, \quad X = x-\epsilon\int_x^\infty \eta\,\mathrm{d}x,$$

this latter transformation being equivalent to writing the equation in a Lagrangian rather than an Eulerian frame. When we eliminate V and ignore terms $O(\epsilon^2)$ and smaller, we are left with essentially the equation that we seek:

$$H_{tt}-H_{XX}-3\frac{\epsilon}{2}(H^2)_{XX}-\frac{\epsilon}{3}H_{XXXX}-\epsilon H_{YY} = 0. \tag{13}$$

This is written in the more conventional form by transforming $H \to -2H/\epsilon$, $(X,t) \to (\epsilon/3)^{1/2}(X,t)$, $Y \to \epsilon Y\surd 3$, to give

$$H_{tt}-H_{XX}+3(H^2)_{XX}-H_{XXXX}-H_{YY} = 0. \tag{14}$$

This is the new two-dimensional Boussinesq equation, from which we recover the classical Boussinesq equation when the dependence on Y is ignored. Equation (14) can be written in bilinear form (see §3), but, as we have mentioned, it does not appear in Hietarinta's (1987) list of completely integrable equations; this has profound consequences for the range of solutions available.

3. Hirota's bilinear form

The KdV equation, (1), 2D KdV equation, (2), and the Boussinesq equation, (4), all possess soliton solutions that are most conveniently represented by setting

$$u = -2\frac{\partial^2}{\partial x^2}\log f \tag{15}$$

and solving for f. The resulting equation for f is, following Hirota (1971), written in terms of the bilinear differential operator, which is defined as

$$\mathrm{D}_t^m \mathrm{D}_x^n (a\cdot b) = \left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^n a(x,t)\, b(x',t')\bigg|_{\substack{x'=x\\ t'=t}} \tag{16}$$

for non-negative integers m and n. Thus, for example, the KdV equation, (1), upon using (15) and (16), becomes (after one integration in x) the bilinear form

$$\mathrm{D}_x(\mathrm{D}_t + \mathrm{D}_x^3)(f\cdot f) = 0.$$

Similarly, the 2D KdV gives the bilinear form

$$(\mathrm{D}_x \mathrm{D}_t + \mathrm{D}_x^4 + \mathrm{D}_y^2)(f\cdot f) = 0 \tag{17a}$$

or

$$f(f_{xt} + f_{xxxx} + f_{yy}) + (3f_{xx}^2 - 4f_x f_{xxx} - f_x f_t - f_y^2) = 0 \tag{17b}$$

(for which we need to include a corresponding dependence on y in (16)), and finally the Boussinesq equation gives

$$(\mathrm{D}_t^2 - \mathrm{D}_x^2 - \mathrm{D}_x^4)(f\cdot f) = 0.$$

Our two-dimensional Boussinesq equation, (14), comprises elements from both the classical Boussinesq equation and the 2D KdV equation; thus we seek solutions which take the form

$$H = -2\frac{\partial^2}{\partial X^2}\log f. \tag{18}$$

With the introduction of the bilinear differential operator, the equation for f becomes simply

$$(\mathrm{D}_t^2 - \mathrm{D}_X^2 - \mathrm{D}_X^4 - \mathrm{D}_Y^2)(f\cdot f) = 0, \tag{19}$$

where we have assumed that $f_X, f_t, f_Y, \ldots \to 0$ as $X \to +\infty$ or $-\infty$. This equation, when written explicitly, is

$$f(f_{tt} - f_{XX} - f_{XXXX} - f_{YY}) - (f_t^2 - f_X^2 - 4f_X f_{XXX} + 3f_{XX}^2 - f_Y^2) = 0. \tag{20}$$

The solutions of these various bilinear equations, which generate soliton solutions via the transformation (15), are obtained by writing

$$\left.\begin{aligned} f &= 1 + \mathrm{e}^{\theta_1} \quad \text{(solitary wave)};\\ f &= 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + A_{12}\mathrm{e}^{\theta_1+\theta_2} \quad \text{(two-soliton)}, \end{aligned}\right\} \tag{21}$$

and so on. The phase function, θ_i, is linear in the independent variables; so for the 2D KdV equation, (17), we find that

$$\theta_i = k_i x + \surd 3 l_i y - \omega_i t + \alpha_i; \quad \omega_i = k_i^3 + 3l_i^2/k_i, \tag{22}$$

where k_i, l_i and α_i are arbitrary real constraints. (The $\surd 3$ here is merely a convenience.) For the two-soliton solution, it can be shown that

$$A_{12} = \frac{(p_1 - p_2)(q_1 - q_2)}{(p_1 + q_2)(q_1 + p_2)},$$

where we have introduced parameters p_i, q_i ($i = 1, 2$) (Freeman 1980) such that

$$k_i = p_i + q_i, \quad l_i = p_i^2 - q_i^2 \quad \text{(and then } \omega_i = 4(p_i^3 + q_i^3)).$$

A special solution of the 2D KdV equation exists when $A_{12} = 0$ (but not with both $p_1 = p_2$ and $q_1 = q_2$, which would recover the solitary-wave solution). Consider $p_1 = p_2$, and $q_1 \neq q_2$; then $A_{12} = 0$ and solution (21) becomes

$$f = 1 + e^{\theta_1} + e^{\theta_2}. \tag{23}$$

This solution represents an interaction with three 'arms' (each locally solitary waves) at infinity; the solution with $A_{12} \neq 0$ possesses four arms describing two waves crossing obliquely. From (23), the three arms are given, at infinity, by

$$1 + e^{\theta_1}, \quad 1 + e^{\theta_2}, \quad 1 + e^{\theta_1 - \theta_2},$$

and this third one satisfies the dispersion relation in (21), with $k_3 = k_1 - k_2$, $l_3 = l_1 - l_2$ and $\omega_3 = \omega_1 - \omega_2$: solution (23) corresponding to the *resonant* interaction of three waves – a resonant triad: see Miles (1977), Freeman (1980). It is readily seen that the resonant solution, (23), satisfies equation (17*b*) by satisfying the two sets of terms in brackets *separately*. This special reduction was exploited by Gibbon, Freeman & Johnson (1978) to obtain solutions of evolution equations which are not completely integrable; what we describe below owes something to this approach.

4. Solutions of the two-dimensional Boussinesq equation

The most natural solution to start our investigation takes the form

$$f = 1 + e^{\theta}, \quad \theta = kX + lY - \omega t + \alpha; \tag{24}$$

direct substitution into equation (19), and using standard properties of the bilinear operator (see e.g. Matsuno, 1984), or directly from (20), yields

$$\omega^2 - k^2 - k^4 - l^2 = 0. \tag{25}$$

The solution (24), with the dispersion relation (25), generates the most general solitary-wave solution of the two-dimensional Boussinesq equation (via the transformation (18)). It is clear that the dispersion relation admits solutions for which $\omega/k > 0$ or $\omega/k < 0$, and so the waves may propagate in either direction.

We now seek a solution, following the familiar structure of (21), in the form

$$f = 1 + e^{\theta_1} + e^{\theta_2} + A_{12} e^{\theta_1 + \theta_2}, \tag{26}$$

where $$\theta_i = k_i X + l_i y - \omega_i t + \alpha_i, \quad i = 1, 2,$$

and $$\omega_i = \epsilon_i (k_i^2 + k_i^4 + l_i^2)^{1/2}, \quad \epsilon_i = \pm 1.$$

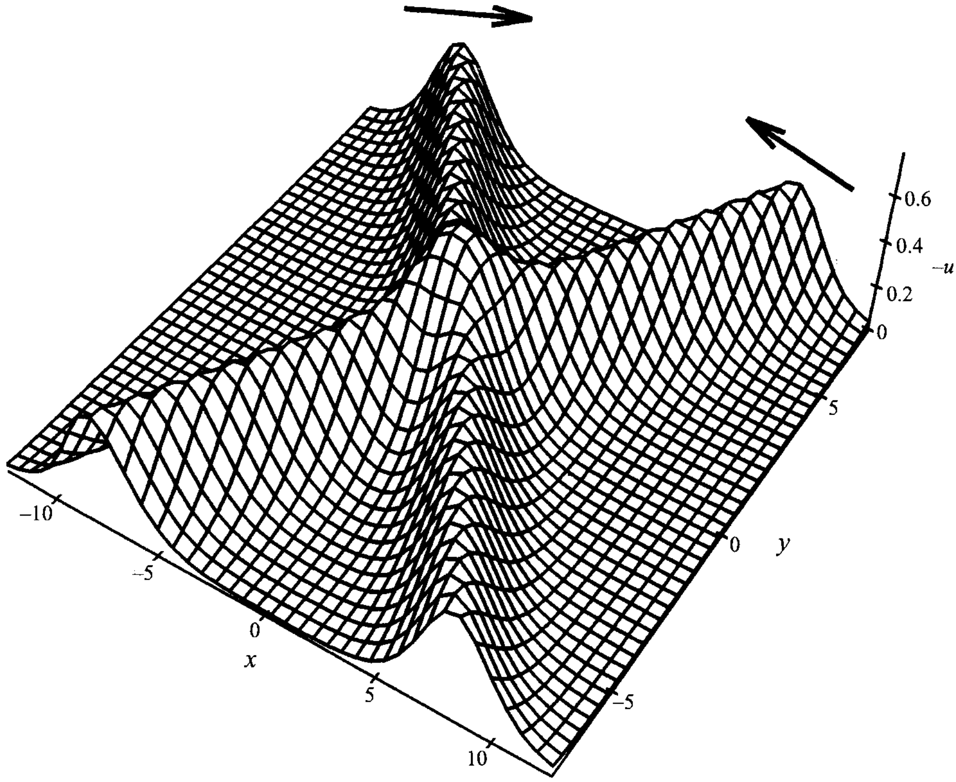

FIGURE 2. An example of a head-on oblique collision of two solitons; $k_1 = l_1 = 1$, $\omega_1 > 0$, $k_2 = 1$, $l_2 = -1$, $\omega_2 < 0$.

It is then fairly straightforward to show that (26) is a solution of (19), for arbitrary values of k_1, k_2, l_1 and l_2, provided

$$A_{12} = \frac{D - 6k_1^2 k_2^2}{D}, \quad D = k_1 k_2 + l_1 l_2 - \omega_1 \omega_2 + k_1 k_2 (2k_1^2 + 3k_1 k_2 + 2k_2^2). \tag{27}$$

The two-soliton solution that is obtained via (18) exists for $A_{12} > 0$ and, because there are no other restrictions on the parameter values, this constitutes the most general two-soliton solution. The structure of the N-soliton solution will be addressed in §6. For the two-soliton solution it is easy to see that we may rewrite the coefficient A_{12}, in (27), as

$$A_{12} = -\left\{ \frac{(\omega_1 - \omega_2)^2 - (k_1 - k_2)^4 - (k_1 - k_2)^2 - (l_1 - l_2)^2}{(\omega_1 + \omega_2)^2 - (k_1 + k_2)^4 - (k_1 + k_2)^2 - (l_1 + l_2)^2} \right\}, \tag{28}$$

which exactly mirrors the result given by Hirota (1973), for the classical Boussinesq equation; the corresponding coefficient for that equation is recovered when we set $l_1 = 0 = l_2$, which merely removes the dependence on Y. The structure of (28) might lead us to believe that the result of Hirota (1973) could be extended to produce the (general) N-soliton solution simply by the inclusion of appropriate terms $(l_i \pm l_j)^2$. However, this would be counter to the predictions of Hietarinta (1987).

For the two-soliton solution in particular, ω_1 and ω_2 may take opposite signs (for both $k_1 > 0$ and $k_2 > 0$, let us say) and consequently the resulting soliton interaction will describe either a head-on or an overtaking oblique collision; an example of a head-on collision is shown in figure 2 (for the choice $k_1 = l_1 = 1$, $\omega_1 > 0$ and $k_2 = 1$, $l_2 = -1$,

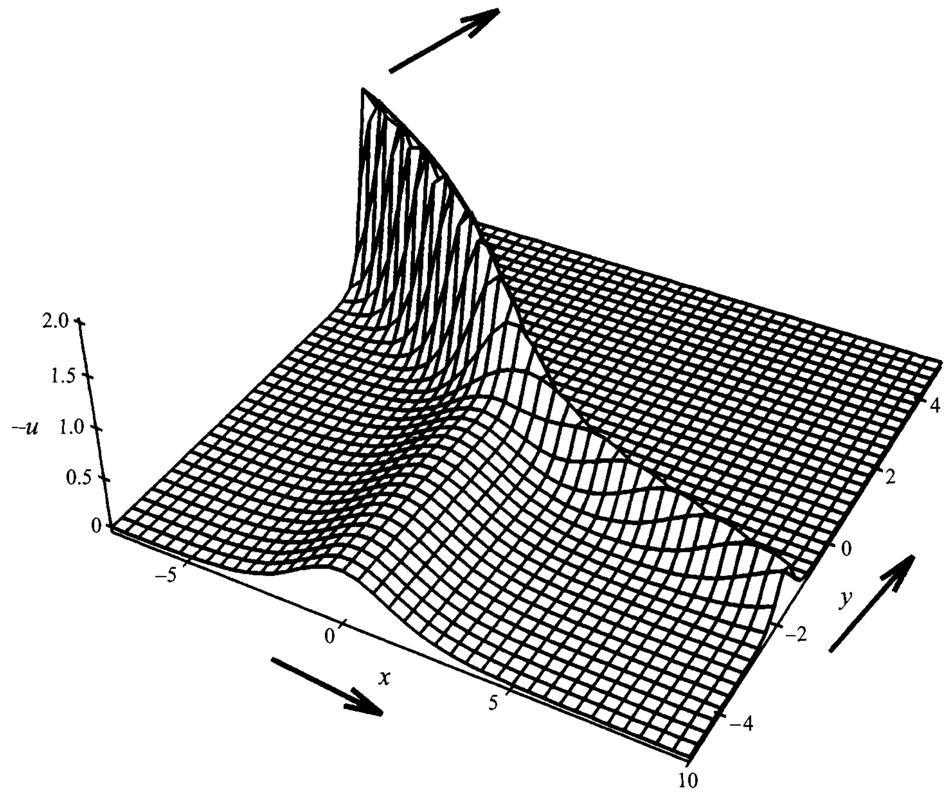

FIGURE 3. An example of a resonant interaction;
$k_1 = 1,\quad l_1 = 0,\quad \omega_1 = \sqrt{2},\quad k_2 = 2,\quad l_2 = \sqrt{30},\quad \omega_2 = 5\sqrt{2}.$

$\omega_2 < 0$). The two-dimensional Boussinesq equation therefore provides an opportunity to model head-on oblique collisions, as we anticipated.

The family of solutions of our new equation also includes resonant interactions obtained, in the two-soliton case, by setting $A_{12} = 0$. Thus, from equation (27), we require that

$$\omega_1 \omega_2 = k_1 k_2 + l_1 l_2 + k_1 k_2 (2k_1^2 - 3k_1 k_2 + 2k_2^2), \tag{29}$$

and when this is satisfied the wave which is described (at infinity) by

$$f = 1 + e^{\theta_1 - \theta_2}$$

satisfies the original dispersion relation with $k_3 = k_1 - k_2$, $l_3 = l_1 - l_2$ and $\omega_3 = \omega_1 - \omega_2$. That this is so is immediately obvious from the alternative version of A_{12} given in (28). Thus the solution represented by

$$f = 1 + e^{\theta_1} + e^{\theta_2}$$

is indeed a resonant interaction (a resonant triad); an example of this is given in figure 3 (for $k_1 = 1$, $l_1 = 0$, $k_2 = 2$, $l_2 = \sqrt{30}$). Corresponding solutions which follow this pattern, namely

$$f = 1 + \sum_{i=1}^{N} e^{\theta_i},$$

can also be obtained (see Gibbon *et al.* 1978), but other solutions based on just the two-soliton structure may also be possible. For example, Freeman (1979) describes how the condition for a resonant solution, coupled with integration over a parameter, leads to a *distributed soliton* of the 2D KdV equation. We now explore what happens when this idea is applied to our two-dimensional Boussinesq equation; we shall see that the results are less satisfactory, although this presents us with an interesting question.

5. Parameterization and the existence of solutions

First we require a suitable parametric representation of the dispersion relation, (25); a convenient one (and the simplest that this author can find) is

$$k = \pm\frac{1}{2}\left(p-\frac{1}{p}\right), \quad l = \pm\frac{1}{8}\left(p^2-\frac{1}{p^2}\right)\left(q-\frac{1}{q}\right), \quad \omega = \pm\frac{1}{8}\left(p^2-\frac{1}{p^2}\right)\left(q+\frac{1}{q}\right), \tag{30}$$

for the two parameters (p, q). The choice of signs is independent in each expression but, for the conventional choice of $k > 0$, we could elect to use the positive sign in k, together with $p > 1$. The special case, $q = 1$, recovers a parameterization of the classical Boussinesq equation.

The condition for a resonant triad, (29), written in terms of the parameters (p_1, q_1) and (p_2, q_2), becomes

$$(p_1^2+1)(p_2^2+1)(q_1-q_2)^2 = 4\frac{q_1 q_2}{p_1 p_2}(p_1-p_2)^2(p_1^2 p_2^2+p_1 p_2+1). \tag{31}$$

This equation is clearly satisfied if $q_1 = q_2$ and $p_1 = p_2$, as we would expect. Furthermore, equation (31) describes the parameter space for which resonant two-soliton solutions exist. Thus, for example, if we solve for q_1/q_2 we find that

$$\frac{q_1}{q_2} = 1+\gamma\pm(\gamma^2+2\gamma)^{1/2}, \quad \gamma = \frac{2(p_1-p_2)^2(p_1^2 p_2^2+p_1 p_2+1)}{p_1 p_2(p_1^2+1)(p_2^2+1)},$$

and so resonant solutions certainly exist whenever both p_i have the same sign; indeed, we can generate any $k_i \geqslant 0$ or $k_i \leqslant 0$ for a suitable value $p_i \geqslant 1$ (see (30)). Hence resonant solutions exist for any pair of wavenumbers (k_1, k_2); the corresponding values of q_1/q_2 are depicted in figure 4, for a range of values of p_1 and p_2. (Note that the expression for γ cannot be written solely as a function of p_1/p_2, although it is symmetric in p_1 and p_2; for $p_1 = p_2$, the only solution is $q_1 = q_2$.)

The distributed-soliton solution described by Freeman (1979), for the 2D KdV equation, might be expected to have its counterpart here. Such a solution takes the form

$$f = 1+\int_{\lambda_1}^{\lambda_2} g(\lambda)\exp\{kX+lY-\omega t\}\,\mathrm{d}\lambda, \tag{32}$$

where k, l and ω satisfy the dispersion relation (25) and which also depend on a parameter λ; $g(\lambda)$ is an arbitrary function. Expression (32) is substituted into equation (20), whereupon the first term in brackets in that equation is immediately zero; the second term gives rise to the double integral

$$\int_{\lambda_1}^{\lambda_2}\int_{\lambda_1}^{\lambda_2} g(\lambda)g(\lambda')(\omega\omega'-kk'-4kk'^3+3k^2k'^2-ll') \times\exp\{(kX+lY-\omega t)+(k'X+l'Y-\omega' t)\}\,\mathrm{d}\lambda\,\mathrm{d}\lambda',$$

where the prime denotes evaluation on $\lambda = \lambda'$. This term is to be zero (for all X, Y and t) if a solution is to exist, and this requires that

$$\omega\omega'-kk'-2(kk'^3+k'k^3)+3k^2k'^2-ll'$$

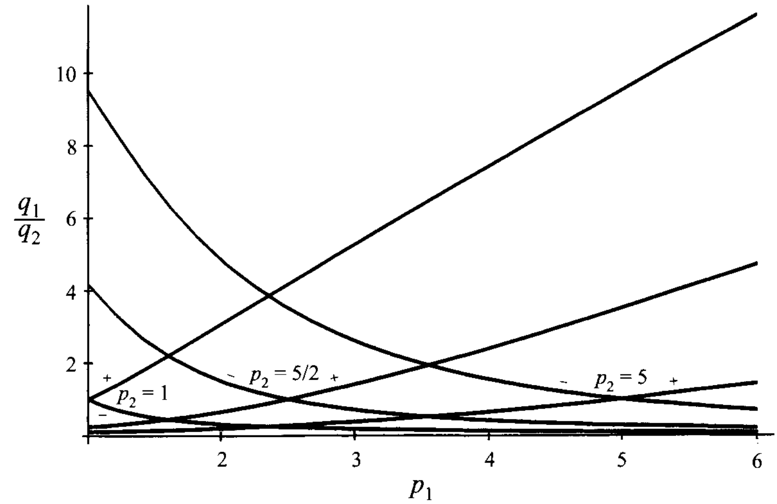

FIGURE 4. Existence of resonant two-soliton interactions; q_1/q_2 against p_1, for various p_2 and showing both branches ($\pm$).

must take a very special form. (Note that we have replaced $4kk'^3$ by $2kk'^3+2k'k^3$, to retain symmetry, which we may always do.) In terms of (p,q), (p',q'), this gives

$$\left(p-\frac{1}{p}\right)\left(p'-\frac{1}{p'}\right)\left\{\frac{(q-q')^2}{pp'qq'}(p^2p'^2+p^2+p'^2+1)-\frac{4(p-p')^2}{p^2p'^2}(p^2p'^2+pp'+1)\right\}, \quad (33)$$

cf. equation (31). What we seek, for an exact solution of the equation, is that either (33) is identically zero or that it takes the form $h(\lambda,\lambda')-h(\lambda',\lambda)$; in either case the double integral is then zero. However, the only available choice that exactly satisfies this condition is

$$p = p' \quad \text{and} \quad q = q',$$

and consequently both p and q are constant: the parameter λ is absent and so the two-dimensional Boussinesq equation does *not* admit a distributed-soliton solution. In the light of Hietrinta (1987), this is not surprising since the existence of a distributed-soliton solution requires conditions stronger than for the existence of a two-soliton solution alone. However, the result does raise the question of how the two-dimensional Boussinesq equation (no distributed soliton) goes over to the 2D KdV equation (which does possess one).

The (linear) dispersion relation for the two-dimensional Boussinesq equation is (see (25))

$$\omega^2-k^2-k^4-l^2 = 0, \quad (34)$$

and for the 2D KdV equation it is

$$\omega k-k^4-l^2 = 0, \quad (35)$$

see equations (2), (17*a*), (22). The connection is made when the two-dimensional Boussinesq problem is interpreted for $\omega \sim \pm k$ and the waves are long, e.g.

$$k = \Delta K, \quad l = \Delta^2 L, \quad \omega = k+\Delta^3\Omega,$$

then, for $\Delta \to 0$, (34) gives

$$2\Omega K-K^4-L^2 = 0,$$

to leading order, which is essentially (35). This approximation is recovered from our parametric representation, (30), when we allow $p \to 1$ and $q \to 1$; we now make this choice, and examine the effect upon the expression (33).

Let us set

$$p = 1 + \delta\lambda, \quad q \sim 1 + \sum_{n=1}^{\infty} \delta^n Q_n, \quad \delta \to 0, \tag{36}$$

and then construct the asymptotic representation of (33) as $\delta \to 0$ (and we observe that, in general, this expression is $O(\delta^4)$). The leading term in (33) contains a factor

$$4(Q_1 - Q_1')^2 - 12(\lambda - \lambda')^2$$

and so we choose

$$Q_1 = \surd 3(\lambda + \alpha_1),$$

where α_1 is fixed i.e. $Q_1' = \surd 3(\lambda' + \alpha_1)$. The next term, $O(\delta^5)$, can also be made zero if we elect to write

$$Q_2 = \tfrac{1}{2}(3 - \surd 3)\,\lambda^2 + 3\alpha_1\,\lambda + \alpha_2,$$

where α_2 is fixed. However, the term that arises at $O(\delta^6)$ can neither be made zero, nor written in the form $h(\lambda, \lambda') - h(\lambda', \lambda)$, for any choice of Q_3. Thus, for the parameters given in (36), a solution which approximates a distributed soliton can be found, where Hirota's bilinear equation, (20), is satisfied with an error of $O(\delta^6)$. The wavenumbers and frequency for this solution are then given by

$$k \sim \pm(\delta\lambda - \tfrac{1}{2}\delta^2\lambda^2), \quad l \sim \pm\delta^2\lambda Q_1, \quad \omega \sim \pm\{k + \tfrac{1}{2}\delta^3(\lambda^3 + \lambda Q_1^2)\},$$

where the first few terms are retained.

6. The non-existence of a general three-soliton solution

We have demonstrated that a general solitary-wave solution and a general two-soliton solution exist, that a resonant triad of waves is also a solution, but that a distributed-soliton solution is not. The next stage in the development is to examine the character of the N-soliton solution. To this end, we first extend the calculation from the two-soliton to the three-soliton solution.

Following the two-soliton solution, (26), we write

$$f = 1 + \sum_{i=1}^{3} e^{\theta_i} + \sum_{\langle i=1 \rangle}^{3} A_{ij}\, e^{\theta_i + \theta_j} + A_{123}\, e^{\theta_1 + \theta_2 + \theta_3}, \tag{37}$$

where $\langle\ \rangle$ denotes that j is taken cyclically with respect to i; the coefficient A_{ij} is defined according to the pattern of (28). Expression (37) is substituted into equation (19) (or (20)); it follows that this f is a solution provided

$$A_{123} = A_{12}\, A_{23}\, A_{31},$$

the familiar result for soliton problems, and provided one further condition is satisfied. The existence of one extra condition (which ensures that the coefficient of the term $e^{\theta_1 + \theta_2 + \theta_3}$ in (20) is zero) implies that the two-dimensional Boussinesq equation is not a soliton equation. The auxiliary condition requires that

$$\sum_{\langle\langle i=1 \rangle\rangle}^{3} \omega_i (k_j\, l_k - k_k\, l_j) = 0, \tag{38}$$

where $\langle\langle\ \rangle\rangle$ denotes that (i, j, k) are taken cyclically. Equation (38) provides a single relation among the six parameters (k_1, l_1), (k_2, l_2) and (k_3, l_3): the expression (37) does

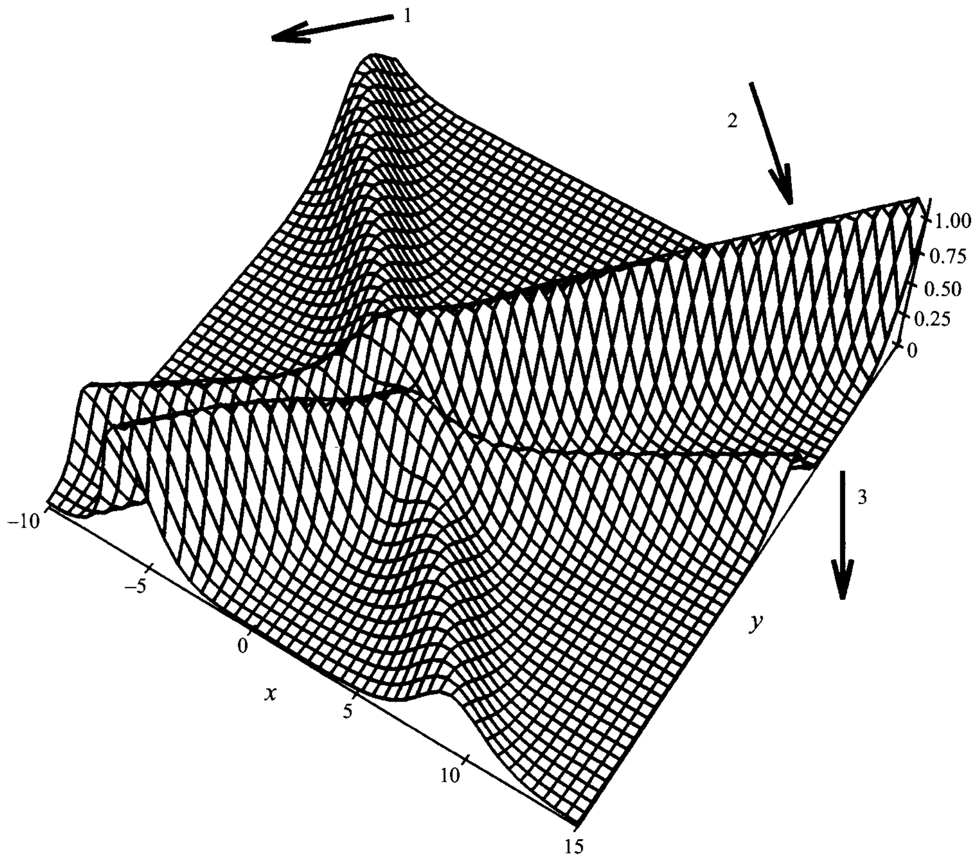

FIGURE 5. An example of a three-soliton interaction;
$k_1 = l_1 = 1,\quad \omega_1 < 0,\quad k_2 = 1,\quad l_2 = -2,\quad \omega_2 > 0,\quad k_3 = 1.5,\quad l_3 \approx -1.74,\quad \omega_3 > 0.$

not lead to the most general form of three-soliton solution for our two-dimensional Boussinesq equation. This result is completely consistent with the analysis of Hietarinta (1987), although constraints like this – and others appear for higher N-soliton solutions – are encountered when special solutions are sought (see e.g. Gibbon *et al.* 1978). In this approach, the number of free parameters decreases as N increases until, beyond a certain $N = N_0$, no solutions of this type exist. (Although the details have not been checked – even with the aid of an algebraic package on a computer, the task is considerable – the pattern evident here, when compared with similar problems, would suggest that $N_0 = 6$; see §7.) Equation (38) is easily solved, for l_3 say, in terms of (k_1, l_1), (k_2, l_2) and k_3. For example, if $k_1 = 1$, $l_1 = 1$, $\omega_1 < 0$, $k_2 = 1$, $l_2 = -2$, $\omega_2 > 0$, with $k_3 = 1.5$ and $\omega_3 > 0$ then $l_3 \approx -1.74$; this three-soliton solution is shown in figure 5. Thus we have a five-parameter family of three-soliton solutions of the two-dimensional Boussinesq equation.

7. Concluding comments

We have summarized how the simplest problem in water waves, describing the propagation of gravity waves, gives rise to a two dimensional Boussinesq equation. This equation has been studied using the Hirota bilinear method and some solutions have been obtained. We have seen that, although this equation can be written in

	Π1	Π2	Π3	Π4	Π5
1	d	A_{ij}	C_{11}	C_{22}	C_{33}
Π1	0	d	A_{ijk}	C_{21}	C_{31}
Π2	d	0	d	A_{ijkl}	C_{32}
Π3	A_{ijk}	d	0	d	
Π4	C_{21}	A_{ijkl}	d	0	d
Π5	C_{31}	C_{32}		d	0

TABLE 1. This grid represents how the various terms required for the N-soliton solution, as they appear in equation (20), are eliminated. Each entry describes the role of the terms obtained by multiplying the column by the row i.e. $\Pi n \times \Pi m$. The notation is as follows: Πn = product $e_i e_j \ldots$ to n terms (cf. equation (37)); 0 = identically vanish; d = eliminated by virtue of the dispersion relation; A_{ij}, etc. = this coefficient in the solution (cf. (37)) is determined; C_{11}, etc. = additional constraint. The heavy lines denote the set of terms used in the two-, three- and four- soliton solutions. (Note that the C_{ij} would be 0 or d in this notation if the equation were completely integrable.)

bilinear form, it does not possess the general three-soliton solution that is the hallmark of a completely integrable equation. The equation does admit general solitary-wave and two-soliton solutions (all these observations being consistent with the work of Hietarinta 1987); resonant triads of interacting waves are also solutions, although the associated distributed soliton is not. Nevertheless, we have demonstrated that a suitable choice of the parameters (namely $p \to 1$, $q \to 1$) does produce a solution which approximates a distributed soliton, and the precise nature of this approximation has been described.

In order to obtain more information about the solutions of this equation (which is no longer a routine exercise in the sense that conventional soliton theory is not directly applicable) we have constructed the three-soliton solution. We have demonstrated that such a solution exists only if an additional constraint is imposed, as anticipated by Hietarinta (1987). An analysis, although not exhaustive, of the problem for larger values of N suggests that, at $N = 4$, two further constraints appear; at $N = 5$, another three, and so on; this pattern is represented in table 1. On this basis, a six-soliton solution exists (but with only two free parameters) and a seven-soliton solution does not. (In keeping with accepted practice, the phase-shift parameters, α_i, are not included in the total of free parameters; although they are all arbitrary, they play an altogether insignificant rôle.) Finally, we observe that the constraint, equation (38), is automatically, satisfied if $l_i = 0$, $i = 1, 2, 3$; since this choice is equivalent to removing the dependence on Y, it confirms that the classical Boussinesq equation possesses a general three-soliton solution (and this same choice, for all l_i, produces the general N-soliton solution of the Boussinesq equation; see Hirota 1973).

We have presented some solutions of the two-dimensional Boussinesq equation;

other special soliton-like solutions may be of interest, even though this equation is not completely integrable. Certainly an area worthy of some exploration is how more general head-on collisions are described by the equation (perhaps using numerical techniques), and their relevance in the study of water waves. That is, for example, how accurately oblique – but nearly parallel, in this approximation – head-on collisions of long gravity waves are modelled by the two-dimensional Boussinesq equation. These are avenues for further study and investigation.

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